Stretching and tilting of material lines in turbulence: The effect of strain and vorticity

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The Lagrangian evolution of infinitesimal material lines is investigated experimentally through three dimensional particle tracking velocimetry (3D-PTV) in quasihomogeneous turbulence with the Taylor microscale Reynolds number $\mathrm{Re}_\lambda=50$. Through 3D-PTV we access the full tensor of velocity derivatives $\partial u_i/\partial x_j$ along particle trajectories, which is necessary to monitor the Lagrangian evolution of infinitesimal material lines l. By integrating the effect on l of (i) the tensor $\partial u_i/\partial x_j$, (ii) its symmetric part s_{ij}, (iii) its antisymmetric part r_{ij}, along particle trajectories, we study the evolution of three sets of material lines driven by a genuine turbulent flow, by "strain only," or by "vorticity only," respectively. We observe that, statistically, vorticity reduces the stretching rate $l_il_js_{ij}/l^2$, altering (by tilting material lines) the preferential orientation between l and the first (stretching) eigenvector λ_1 of the rate of strain tensor. In contrast, s_{ij}, in "absence" of vorticity, significantly contributes to both tilting and stretching, resulting in an enhanced stretching rate compared to the case of material lines driven by the full tensor $\partial u_i/\partial x_j$. The same trend is observed for the deformation of material volumes.

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I. INTRODUCTION

Turbulent mixing and dispersion are among the most intriguing and difficult problems for the fluid mechanics community. The physical mechanisms underlying mixing processes are naturally addressed in a Lagrangian manner, i.e., following the evolution of material fluid elements, as lines, surfaces, and volumes. These material elements are Lagrangian objects with zero diffusivity passively driven by the turbulent flow, in the sense that the field of velocity derivatives is only one-way coupled with the material elements. A number of numerical studies have been devoted to the investigation of statistical properties of material elements in a turbulent flow field in the last decades ([1–6], among others). Recently, experimental studies on infinitesimal material elements became possible, thanks to the three dimensional particle tracking velocimetry (3D-PTV) experimental technique, allowing for sufficiently accurate estimation of the Lagrangian evolution of the full tensor of velocity derivatives [7,9]. Batchelor [8] provided theoretical predictions on material line stretching and surface stretching, based on the analysis of the evolution of infinitesimal material line elements, considered as the smallest segments that construct any line of finite length. Since then, the studies were largely performed on the equation $dl_i(t)/dt=\partial u_i/\partial x_j l_j(t)$ (for instance [2,10]). $l_i(t)$ is the infinitesimal material line element, and the tensor of velocity derivatives $\partial u_i/\partial x_j$ $(i,j=1,2,3$ denote vector components) is evaluated along the trajectory. An analog approach (see [2,11] and the following sections) describes the deformation of an initially spherical infinitesimal material volume into an ellipsoid, induced by the turbulent flow. The evolution of infinitesimal material elements is driven by the velocity gradient tensor, attached to the moving fluid particle. It is of interest to elucidate the separate effects of the symmetric (i.e., rate of strain, s_{ij}) and antisymmetric (i.e., vorticity, ω) parts of the velocity gradient tensor. Currently, the understanding is that an infinitesimal material line is tilted by the vorticity field and stretched or compressed along the eigenvectors (λ_k) of the rate of strain tensor which, in turn, rotates with an angular velocity comparable in magnitude with vorticity ([2,10]). Indeed, the rotation of the eigenframe of the rate of strain tensor, often referred to as "nonpersistence of strain" (e.g. [2,9,10,12]) contributes also to the tilting of material lines. An open question is then how much the field of strain and the field of vorticity contribute respectively to the Lagrangian evolution of the orientation of material lines and, similarly, whether strain and vorticity statistically act in the same direction, or against each other. The change of orientation of material lines, denoted here as tilting, is a key quantity since it relates to the alignments of l with the eigenframe of strain, (l,λ_k), and consequently to the rate of stretching $l_il_js_{ij}/l^2$, since $l_il_js_{ij}/l^2=\Lambda_1\cos^2(l,\lambda_1)+\Lambda_2\cos^2(l,\lambda_2)+\Lambda_3\cos^2(l,\lambda_3)$. In order to answer these questions, we adopt the idea of an artificial numerical experiment in [2], in which the numerical investigation focused on special material lines initially oriented along the most stretching eigenvector λ_1. Enhanced mean stretching was observed for those material lines that are driven by "strain only," compared to the material lines driven by the velocity gradient tensor $\partial u_i/\partial x_j$. At first, we confirm the results in [2] by conducting a similar experiment on our 3D-PTV data ([7,9]). We then extend the analysis to a set of randomly oriented material lines and we proceed to the analysis of the separate effect of vorticity and strain on the orientation and on the tilting of infinitesimal material lines. Further, we extend the study to the deformation of material volumes. As we

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will explain in Sec. II, the method allows for neglecting only the "direct" effect of vorticity or strain. The strong coupling between vorticity and strain is such (see [13] and references therein) that the magnitude of strain, as well as the orientation of its eigenframe, are not independent of vorticity, and vice versa. Therefore in this numerical experiment, even when the effect of vorticity on the evolution of material lines is neglected, ω is still influencing the strain field (and its eigenframe λ_k) and, thus, vorticity is still "indirectly" participating in the evolution of material elements. Since both vorticity and strain have been obtained in a real turbulent flow, they are genuinely coupled. Despite this fact, our main goal is to decouple and investigate the effects of vorticity and strain on the stretching or tilting of material lines and on the deformation of material volumes.

II. METHOD

In the present section we provide a brief summary of the experimental apparatus and the description of the procedure leading to the estimate of the Lagrangian evolution of infinitesimal material lines. The flow domain was a rectangular glass tank of $12\times12\times14$ cm^3. Two 2×2 arrays of rare-earth strong permanent magnets (diameter of 4.2 cm) were positioned at the two opposite side walls. We used as a fluid a saturated copper sulphate solution ($CuSO_4$) with conductivity of 16.7 mS cm^{-1}, density of 1050 kg m^{-3}, and viscosity of 1.2×10^{-6} m^2 s^{-1}. The flow was forced electromagnetically by Lorentz forces $\mathbf{f_l}=\mathbf{j}\times\mathbf{B}$, where $\mathbf{j}$ is a current density of 70 A m^{-2} and $\mathbf{B}$ is the magnetic field. The oscillating swirling motions forming in the proximity of each magnet, merged in the center of the tank, where quasihomogeneous turbulent flow was achieved. The flow was seeded with neutrally buoyant 40 μm polystyrene particles, enlightened by an expanded beam of 25 W Ar-Ion continuous wave laser. Particle images in an observation volume of approximately 4.5 cm^3 were recorded by four synchronized cameras at a frame rate of 50 Hz. The three dimensional location of each particle was, first, reconstructed from the particle images of at least three cameras, and then linked to the correspondent particle in the consecutive frame. The details of the experimental method, the estimation of velocity derivatives, the validation of the estimated quantities, and the discussion of the experimental errors is provided in [7]. On average, 600 particles were tracked in each frame for 100 s of total observation time. The Kolmogorov time and length scales were estimated as 0.25 s and 0.55 mm, respectively and the r.m.s. velocity was approximately 6 mm s^{-1}. Defining at each point 10 randomly oriented material lines, the total number of material lines in this analysis is of the order of 10^6.

The evolution of infinitesimal material lines is governed by the equation (e.g., [2,11])

$$l(t)=B(t)l(0), \quad (1)$$

where the matrix B evolves in time according to

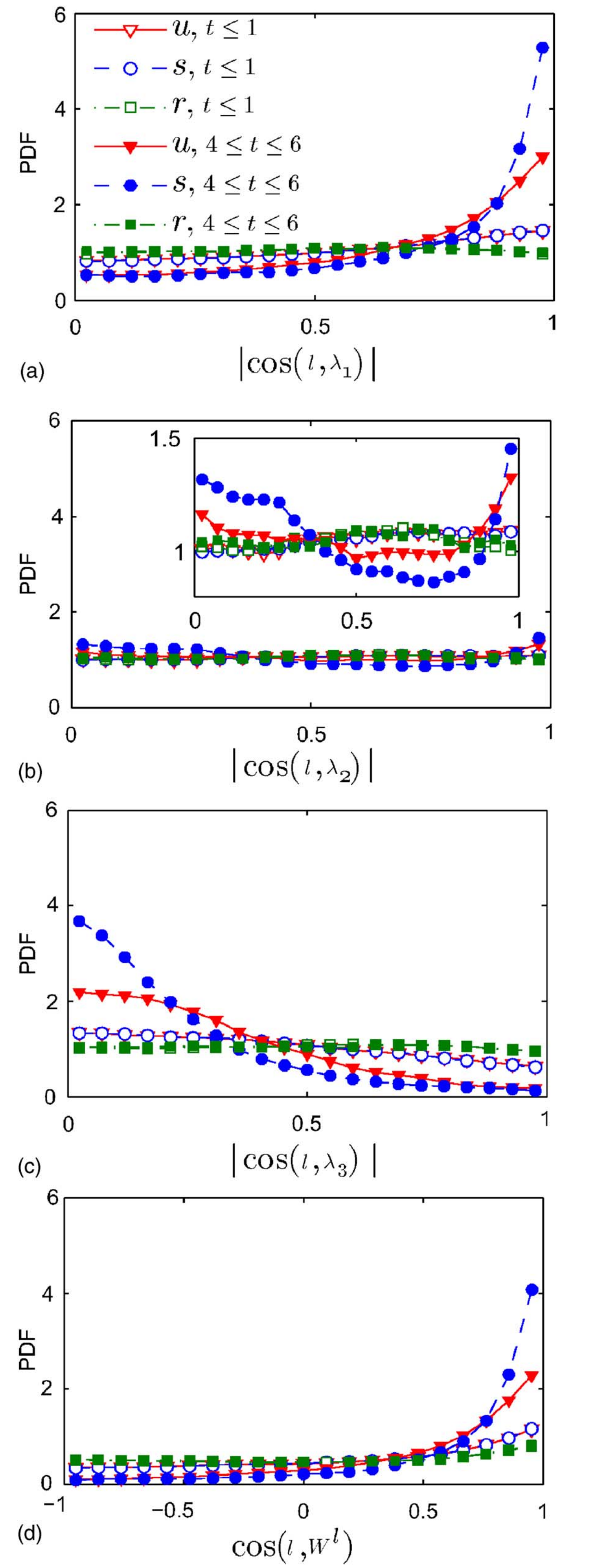

FIG. 1. (Color online) PDF of the (l,λ_k) k=1,2,3; (a)–(c) and (l,W^l); (d) alignment for l governed by the full tensor $\partial u_i/\partial x_j$ ($\bigtriangledown$), its symmetric part s_{ij} ($\bigcirc$) and its antisymmetric part r_{ij} ($\square$). Note that the initial orientation is random.

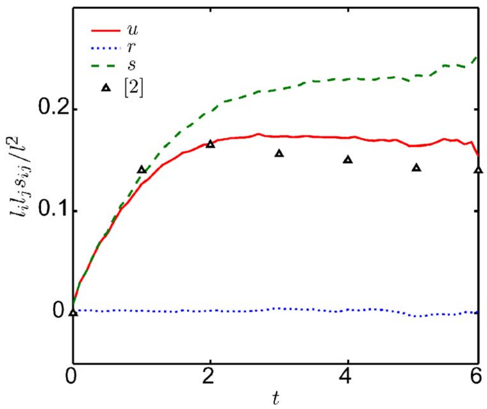

FIG. 2. (Color online) Mean Lagrangian evolution of $l_i l_j s_{ij}/l^2$ for material lines, governed by the full tensor $\partial u_i/\partial x_j$ (u), its symmetric part (s) and its antisymmetric part (r). ($\triangle$) indicate numerical results of Girimaji and Pope [2].

$$\frac{d}{dt}B = h(t)B(t), \quad h(t) = \left(\frac{\partial u_i}{\partial x_j}\right)_t B(0) = I, \tag{2}$$

where I is an identity matrix. We can perform three numerical experiments on the data obtained through 3D-PTV [7]. We compare the statistical behavior, in terms of preferential orientation, tilting and stretching rates, of infinitesimal material lines l, with the correspondent behavior of two other distinct sets of material lines, l_s and l_r. The set of l is derived by integrating along the particle trajectory the full tensor $h(t)=(\partial u_i/\partial u_j)_t$, while the set of l_s is derived by integrating the symmetric part $s_{ij}(t)$ and l_r by integrating the antisymmetric part $r_{ij}(t)$ only

$$l_s(t) = B_s(t)l_s(0), \quad l_r(t) = B_r(t)l_r(0) \tag{3}$$

where $l(0)\equiv l_s(0)\equiv l_r(0)$ is an initially randomly oriented set of material lines. B_s and B_r evolve according to the following equations:

$$dB_s/dt = h(t)B_s(t), \quad h(t) = s_{ij}(t), \tag{4}$$

$$dB_r/dt = h(t)B_r(t), \quad h(t) = r_{ij}(t). \tag{5}$$

The main point of this approach is to follow the evolution of material lines in a measured turbulent flow but, as if either ω or s_{ij} is "missing," in the sense of not contributing to the evolution of l_s or l_r, repsectively. From the technical point of view, the proposed method is a valid tool for studying, in turbulent flows, the effect of strain and enstrophy on the behavior of material elements, in addition to the common procedure of conditional sampling based on the magnitude of strain and vorticity. Through the rest of the paper, we deal with dimensionless quantities: the velocity gradient tensor $\partial u_i/\partial x_j$ and time t are normalized by the Kolmogorov time scale τ_η.

III. RESULTS

The three sets of material elements, l, l_s, and l_r, were observed to evolve differently. We see from Fig. 1(a) how l_s develop a stronger predominant alignment with the most stretching eigenvector λ_1 compared to l and l_r. Since l_r rotate with vorticity, it is no surprise that they remain randomly oriented. In Fig. 1(b) we see how the PDFs for $\cos(l,\lambda_2)$ remain flat for all times and all three sets. Consistently with Fig. 1(a), and most pronounced for l_s, material lines tend to evolve to a predominantly λ_3-normal orientation, as shown in Fig. 1(c). The fact that especially l_s are predominantly stretched, rather than compressed, is reflected in the probability density function (PDF) for $\cos(l,W^l)$, where $W^l=l_j s_{ij}$ is the material line stretching vector, Fig. 1(d). The net result of these alignments with respect to time evolution of the mean material line stretching rate $l_i l_j s_{ij}/l^2$ is illustrated in Fig. 2 and compared to the numerical results of Girimaji and Pope [2] for randomly oriented material lines. The stretching rate for l_s is clearly larger than for l and the stretching rate for l_r remains zero. We then estimate the effect of $\partial u_i/\partial x_j$, s_{ij}, and r_{ij} on the tilting of material lines, $\eta_l^2 \equiv D(l_i/l)/Dt \cdot D(l_i/l)/Dt$ through the equation

$$\eta_l^2 = \frac{(W^l)^2}{l^2} - \left\{\frac{l_i l_j s_{ij}}{l^2}\right\}^2 + \frac{l_j s_{ij}(\omega\times l)_i}{l^2} + \frac{(\omega\times l)^2}{4l^2}. \tag{6}$$

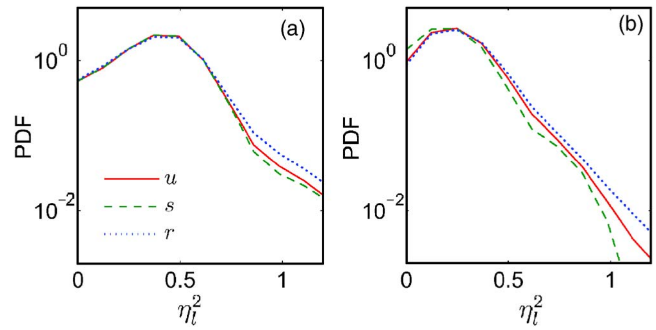

FIG. 3. (Color online) PDF of η_l^2 for $l(t<1\tau_\eta)$; (a) and $l(t>4\tau_\eta)$; (b) governed by the full tensor (solid line), its symmetric part (dashed line) and its antisymmetric part (dotted line).

In Fig. 3 we show that the strain field is effectively tilting material lines, almost as much as the full tensor. However, while l_s are tilted only by the strain field such to keep pursuing the nonpersistent λ_1 direction, l are also tilted by

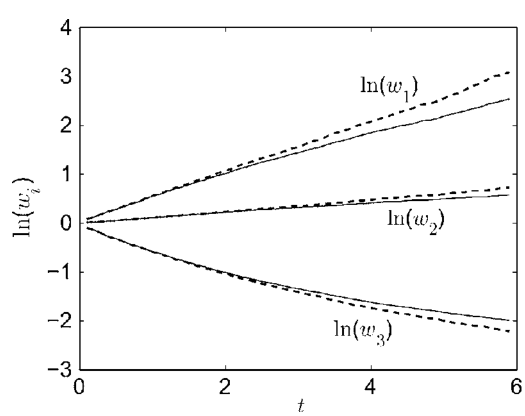

FIG. 4. Mean Lagrangian evolution of the eigenvalues w_i of $W=BB^T$ (full lines) and of $W=B_sB_s^T$ (dashed lines).

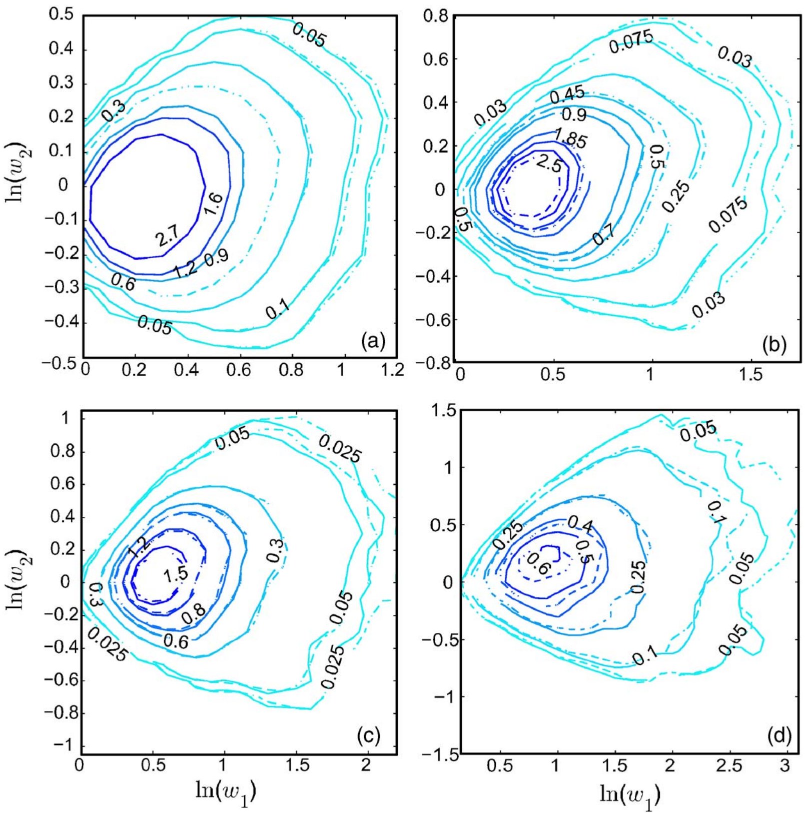

FIG. 5. (Color online) Joint PDF of $\ln(w_1)$, $\ln(w_2)$, where w_k are the eigenvalues of $W=BB^T$ (full lines) and of $W=B_sB_s^T$ (dashed lines). (a) $1\tau_\eta$, (b) $2\tau_\eta$, (c) $3\tau_\eta$, and (d) $4\tau_\eta$.

the vorticity field. The overall picture suggests that vorticity reduces (in case of l) and even neutralizes (in case of l_r) the predominant alignment of l with λ_1 that develops over time due to strain kinematics. An additional question is whether the enhanced stretching of material lines l_s, under the effect of strain, leads to an enhanced deformation of an infinitesimal fluid volume. Following the analysis in [2], we start with a fluid sphere at $t=0$. In a turbulent flow this sphere develops into an ellipsoid with (generally different) axes ratios $a:b:c$ changing in time according to the evolution of the Cauchy Green tensor $W=BB^T$, with B defined in Eq. (2). It follows that $a:b:c=w_1^{1/2}:w_2^{1/2}:w_3^{1/2}$, where w_k are the eigenvalues of the W tensor. Since $B(0)=I$ [Eq. (2)], $W(0)=I$, and $w_k(0)=1$. Due to the conservation of the initial volume, at any time t it has to be satisfied that $w_1(t)\cdot w_2(t)\cdot w_3(t)=w_1(0)\cdot w_2(0)\cdot w_3(0)=1$, leading to $\langle\ln(w_1)\rangle+\langle\ln(w_2)\rangle+\langle\ln(w_3)\rangle=0$. We extend our analysis by studying the evolution of another set of material volumes derived by using B_s [Eq. (4)] instead of B. The Lagrangian evolution of w_k is shown in Fig. 4. Note that we omit the case of material volumes driven by "vorticity only," since there is no deformation but only a rigid body rotation. The volume deformation appears to be enhanced under the effect of "strain only," compared to the case of genuine turbulence. This enhancement is consistent with the effect observed on the stretching of material lines. However, the contours of the joint PDF of w_1 and w_2, in Fig. 5, have similar shape implying that the relative distribution of volumes deformed into *cigars* or *pancakes* [2], is not altered by the lack of the vorticity effect. We quantitatively confirm this in Fig. 6, by observing that the ratio w_2/w_1, at different stages of evolution, is not affected.

IV. CONCLUSIONS

The effect of strain and vorticity on the evolution of material lines and deformation of material volumes is investigated via a numerical experiment performed on the 3D-PTV data obtained in homogeneous turbulence [7]. Following the approach in [2], we estimate the Lagrangian evolution of infinitesimal material lines and volumes, under three different conditions, namely under the effect of vorticity only, strain only, and in a genuine turbulent flow, for comparison. The proposed analysis confirms that, in a turbulent flow, both the vorticity field and the strain field contribute to the tilting

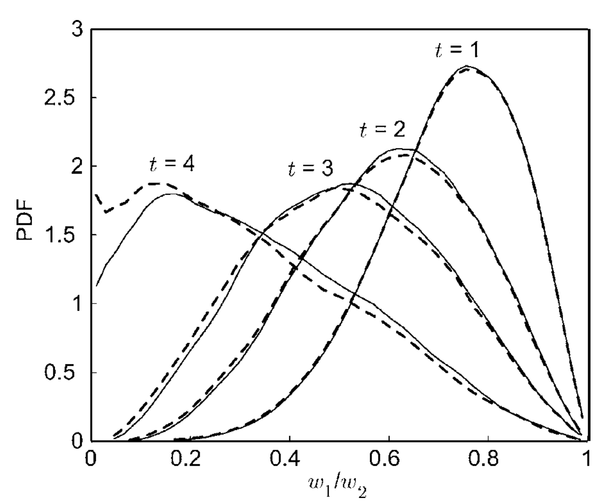

FIG. 6. PDF of the ratio of the eigenvalues w_1/w_2 of $W=BB^T$ (full lines) and of $W_s=B_sB_s^T$ (dashed lines).

of material lines. The strain field, by stretching material lines along the direction of the first eigenvector λ_1 (which is rotating with the strain eigenframe as shown in [2], [9], [10]), contributes also to tilting. Our findings can be summarized in two points: (i) material lines and volumes exhibit an enhanced stretching rate when the direct influence of vorticity is "missing." This suggests that vorticity is statistically acting against stretching. In particular the contribution to the tilting of l of vorticity and the contribution of the nonpersistently oriented strain are statistically opposing each other (e.g., the alignment l, λ_1 appears to be weaker than the alignment l_s, λ_1); (ii) the Lagrangian evolution of fluid volumes into "cigar" and "pancakes" [2] is not affected by the presence or absence of the effects of the vorticity field. It is a common approach to consider the processes of mixing as a combination of stretching and folding processes. In the results presented here, there is an indication that vorticity, statistically, inhibits the stretching process. It still remains to be investigated experimentally how also folding is governed by the field of velocity derivatives.

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